Singular Solutions and Integral Transport Theory

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Abstract

The integral transport equation clearly indicates that, in a void region, the angular flux is constant along a characteristic. Yet simple physical arguments can be used to demonstrate that angular flux solutions having a delta-function angular dependence can appear to be non-constant along a characteristic. Furthermore, a naive application of the integral transport equation to problems of this type can result in erroneous results. Using some simple examples, we resolve this apparent contradiction, and give some rules for properly applying the integral transport equation to problems with angular delta-function solutions.

1 Introduction

A fundamental principle of transport theory states that the angular flux in a void is constant along a characteristic. This principle is easily demonstrated via the integral transport equation.[2] Nonetheless, simple physical arguments can be used to show that there are singular angular flux solutions in voids that appear to vary along a characteristic. Furthermore, a naive application of the integral transport equation to problems with such solutions can yield erroneous results. The purposes of this paper are to give insight into the nature of angular solutions with delta-function angular dependencies, and to show that correct solutions are obtained from the integral transport equation if one simple rule is followed.

The remainder of this note is organized as follows. First we derive the singular solution to a simple problem using nothing more than fundamental transport concepts. Next we confirm this solution using the integrodifferential form of the transport equation. We then demonstrate a naive use of the integral transport equation that yields an incorrect solution for the example problem, followed by an approach that yields the correct solution regardless of the equation being solved. In the later case, we make use of solutions for 1-D purely absorbing spheres and cylinders originally given by Lathrop.[1] Finally, we consider a second example problem and obtain a singular solution using the correct approach together with the integral transport equation. The simplicity of the problem makes it possible to gain considerable insight into the nature of the corresponding angular flux solution.

For completeness, Lathrop's solutions for purely absorbing spheres and cylinders with arbitrary non-singular incident fluxes are derived in the Appendix. Solutions are also derived for singular incident fluxes.

2 An Example Problem

The 1-D spherical-geometry transport equation is[2]:

$$\mu\frac{\partial}{\partial r}\psi + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\psi + \sigma_t\psi = \sigma_s\phi + q\,, \tag{1a}$$

where $\mu = \cos(\theta)$ is the direction variable, r is the radial coordinate, ψ is the angular flux, σ_t is the total macroscopic cross section, σ_s is the macroscopic scattering cross section, q is the inhomogeneous source, and ϕ is the scalar flux:

$$\phi = \frac{1}{2}\int_{-1}^{1} \psi(\mu)\, d\mu\,. \tag{1b}$$

The problem we first consider consists of a voided sphere of radius, a, with an incident flux on its outer boundary defined as follows:

$$\psi(a,\mu) = 2\delta(\mu+1) \quad \text{for } \mu < 0. \tag{2}$$

Note that the incident flux is normal to the spherical surface and corresponds to a unit incident current.

3 The Analytic Solution via Basic Concepts

Using basic transport concepts, we find that the total number of particles entering the system is

$$-4\pi a^2 \left[\frac{1}{2}\int_{-1}^{0} \mu\psi(a,\mu)\, d\mu\right] = 4\pi a^2 \,. \tag{3}$$

All of the particles that enter the sphere pass through the origin and then exit the sphere with the direction $\mu = +1$. Taking the radial symmetry of the problem into account, it follows that the solution on the surface of the sphere is

$$\psi(a,\mu) = 2\left[\delta(\mu+1) + \delta(\mu-1)\right] \,. \tag{4}$$

Consider an inner sphere of arbitrary radius $0 \le r \le a$. The particles that enter and exit the outer sphere similarly enter and exit the inner sphere without change of direction. Thus the angular flux solution for the inner sphere must have the same delta-function dependence as the solution for the outer surface of the sphere. A particularly convenient way to express this solution is as follows:

$$\psi(r,\mu) = \phi(r)\ \left[\delta(\mu+1) + \delta(\mu-1)\right] \,, \tag{5}$$

where $\phi(r)$ is the scalar flux. To obtain an explicit expression for $\phi(r)$, we use the fact that the number of particles entering the inner sphere must be equal to the the number entering the outer sphere:

$$-4\pi r^2\phi(r) \left[\frac{1}{2}\int_{-1}^{0} \mu\, \delta(\mu+1)\, d\mu\right] = -4\pi r^2\, \phi(r)/2 = -4\pi a^2 \,. \tag{6}$$

Solving Eq. (6) for $\phi(r)$, we obtain

$$\phi(r) = 2\,(a^2/r^2)\,. \tag{7}$$

Substituting from Eq. (7) into Eq. (5), we obtain the desired angular flux solution

$$\psi(r,\mu) = 2(a^2/r^2)\;[\delta(\mu+1)+\delta(\mu-1)]\;, \tag{8}$$

4 Application of the Integral Transport Equation

The integral transport equation[2] takes on a particularly simple form in a sourceless void region. Specifically,

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \psi(\overrightarrow{r} - s_b\,\overrightarrow{\Omega}, \overrightarrow{\Omega})\,, \tag{9}$$

where s_b is the distance upstream along the characteristic from the point $\overrightarrow{r}$ to the outer boundary of the void region. Equation (9) simply states that the angular flux solution along the characteristic is constant and equal to the incident angular flux along that characteristic. If we assume that that $\psi(\overrightarrow{r} - s_b\,\overrightarrow{\Omega}_0, \overrightarrow{\Omega}) = \delta(\overrightarrow{\Omega} - \overrightarrow{\Omega}_0)$, and substitute this expression into Eq. (9), we get

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}_0) = \delta(\overrightarrow{\Omega} - \overrightarrow{\Omega}_0)\,. \tag{10}$$

In 1-D spherical geometry, particles in the directions $\mu = \pm 1$ do not change direction as they stream, therefore Eq. (9) restated for $\mu = -1$ is

$$\psi(r,-1) = \psi(a,-1)\,. \tag{11}$$

Substituting from Eq. (2) into Eq. (11), we get

$$\psi(r,\mu) = 2\,\delta(\mu+1) \quad \text{, for } \mu \text{ about } -1, \tag{12}$$

Whereas Eq. (11) is correct if $\psi(a,\mu)$ is non-singular about $\mu=-1$, Eq. (12) is clearly wrong. Thus one cannot simply insert a delta-function incident flux into the integral transport equation.

5 Solution via the Integrodifferential Equation

In this section we show that the correct solution for our example problem is obtained by directly inserting the delta-function incident flux into the 1-D integrodifferential form of the spherical-geometry transport equation. In particular, in a void, Eq. (1a) reduces to

$$\mu\frac{\partial}{\partial r}\psi + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\psi = 0\,. \tag{13}$$

Substituting From Eq. (5) into Eq. (13), we get

$$\mu\frac{\partial}{\partial r}\left\{\phi(r)\left[\delta(\mu+1)+\delta(\mu-1)\right]\right\} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\left\{\phi(r)\left[\delta(\mu+1)+\delta(\mu-1)\right]\right\} = 0\,. \tag{14}$$

To deal with the delta-function derivatives associated with the angular derivative term in Eq. (14), we note that the derivative of a delta-function extracts the negative of the derivative of a function under integration, i.e., that

$$\int_{-\epsilon}^{+\epsilon}\frac{\partial\delta(x)}{\partial x}f(x)\,dx = \Big|_{-\epsilon}^{+\epsilon}\,\delta(x)f(x) - \int_{-\epsilon}^{\epsilon}\delta(x)\frac{\partial f}{\partial x} = -\left.\frac{\partial f}{\partial x}\right|_{x=0}\,, \tag{15}$$

Integrating Eq. (14) over the negative directions, we obtain

$$-\frac{\partial}{\partial r}\phi(r) - \frac{2}{r}\phi(r) = 0\,. \tag{16}$$

Integrating Eq. (14) over the positive directions and then multiplying the resulting equation by -1 also yields Eq. (16).

The general solution to Eq. (16) is

$$\phi = \frac{C}{r^2}\,, \tag{17}$$

where C is a constant. By requiring that $\phi(a) = 2.0$, we get $C = 2a^2$, which is consistent with Eqs. (7) and (8).

It is interesting to note that if $\frac{\partial \psi}{\partial \mu}$ is bounded at $\mu = -1$, taking the limit of Eq. (13) as $\mu \to -1$ yields

$$-\frac{\partial}{\partial r}\psi(r,-1) = 0\,. \tag{18a}$$

Proceeding analogously for $\mu = +1$, we get

$$\frac{\partial}{\partial r}\psi(r,+1) = 0\,. \tag{18b}$$

Note that this result is consistent with the principle that the angular flux in a void is constant along the characteristic. Note also that the angular flux is not singular in this case, which is the key point.

6 Lathrop's Solution

Lathrop has given an analytic solution for a purely absorbing sphere of radius, a, with an arbitrary incident flux on the boundary, $\psi_b(\mu)$.[1] In this section, we use this more general solution to obtain the solution to our example problem. For instance, taking the limit of Lathrop's solution as the absorption cross-section goes to zero, we obtain his corresponding void solution:

$$\psi(r,\mu) = \psi_b(\mu)\big|_{\mu=\mu'} \ , \tag{19a}$$

where

$$\mu' = -\sqrt{1-(r^2/a^2)(1-\mu^2)} \, . \tag{19b}$$

As shown in the Appendix, this expression is determined by the geometry of the straight-line characteristics in the sphere. It states that the boundary flux contributions to the angular flux at (r,μ) come from the direction μ' on the boundary.

We next assume that $\psi_b(\mu) = 2\delta(\mu+1)$, and substutute this expression into Eq. (19a) to obtain:

$$\psi(r,\mu) = 2\delta\left[1-\sqrt{1-(r^2/a^2)(1-\mu^2)}\right] \tag{20}$$

Because it has a complicated argument, is convenient to re-express the delta-function in Eq. (20). In particular, consider a delta-function of the form, $\delta[f(x)]$, where $f(x)$ is an arbitrary but complicated function. A simplification can be effected by first transforming the integration variable from x to $y = f(x)$:

$$\int \delta\left[f(x)\right] h(x)\, dx = \int \delta(y) h(y) \left|\frac{dy}{dx}\right|^{-1} dy = \left[h(y)\left|\frac{dy}{dx}\right|^{-1}\right]_{y=0} . \tag{21}$$

Assuming that $f(x)$ has a single root, x_r, it is clear that

$$\int \delta(x - x_r) h(x) \left|\frac{dy}{dx}\right|^{-1} dx = h(x_r) \left|\frac{dy}{dx}\right|_{x=x_r}^{-1} . \tag{22}$$

Comparing Eqs. (21) and (22), we find that

$$\delta\left[f(x)\right] = \left|\frac{df}{dx}\right|_{x=x_r}^{-1} \delta(x - x_r) . \tag{23}$$

If $f(x)$ has multiple roots, e.g., $\{x_{r1}, x_{r2}, \ldots\}$, then Eq. (23) becomes

$$\delta\left[f(x)\right] = \sum_i \left|\frac{df}{dx}\right|_{x=x_{ri}}^{-1} \delta(x - x_{ri}) . \tag{24}$$

The argument of the delta-function in Eq. (20) is zero for $\mu = \pm 1$. Using Eq. (24), we find that

$$\delta\left[1 - \sqrt{1 - (r^2/a^2)(1 - \mu^2)}\right] = (a^2/r^2)\left[\delta(\mu + 1) + \delta(\mu - 1)\right] . \tag{25}$$

Substituting from Eq. (25) into Eq. (20), we obtain Eq. (8).

Lathrop actually used integral transport theory to obtain his general solution. This suggests that the integral transport equation does yield the correct solution with delta-function incident fluxes if you first obtain a solution for an arbitrary non-singular incident flux, and then assume a delta-function angular dependence for the incident flux. The key is that you must use the integral transport equation to obtain a solution *before* making the delta-function assumption, rather than first making the delta-function assumption and then using the integral transport equation to obtain a solution.

The full derivations of Lathrop's solutions for both 1-D purely-absorbing spheres and 1-D purely-absorbing cylinders are given in the Appendix. These general solutions are also used to obtain solutions for delta-function incident fluxes.

7 A Second Example Problem

In this section we consider a second example problem that is very similar to our first example problem, but is sufficiently simple to yield considerable insight into why one cannot simply insert delta-function sources into the integral transport equation and expect to get correct results. To make the problem simple, we consider transport in 2-D $X-Y$ geometry with the particle directions limited to the unit disk, i.e., lying in the $X-Y$ plane. We seek the scalar flux solution at an arbitrary point within a voided disk of radius a, having a constant normally-incident flux at each point on the outer surface of the disk. The incident flux distribution is illustrated in Fig. 1. The spatial coordinates used for this problem are the standard cylindrical coordinates, (ρ, θ), as illustrated in Fig. 2. There is a single directional coordinate used for this problem, ω, which is also illustrated in Fig. 2. Only one directional variable is required since all of the directions lie in the $X-Y$ plane. Note that $\omega = \pi$ is always directed toward the origin. Thus the directional coordinate system is analogous to that used in 1-D spherical geometry. This is appropriate since the solution to this problem is radially symmetric (independent of θ) using this directional coordinate system. The scalar flux is defined in this coordinate system as follows:

$$\phi = \frac{1}{2\pi} \int \psi \, d\omega; \, , \tag{26}$$

and the incident flux, normalized to a unit incident current, is given by:

$$\psi(a, \theta, \omega) = 2\pi \, \delta(\omega - \pi) \quad , \text{ for all } \theta. \tag{27}$$

One can use the same fundamental principles and arguments used for the first example problem to show that the solution to this problem is

$$\psi(\rho, \omega) = 2(a/r) \left[\delta(\omega - \pi) + \delta(\omega)\right] \, . \tag{28}$$

The integral transport equation for this 2-D system is identical to that for 3-D systems. The directions are simply restricted to the unit disk in the $X - Y$ plane. In analogy with the spherical-geometry case, particles in the directions $\omega = \pi$ and $\omega = 0$ do not change their direction as they stream, so Eq. (9) can be re-expressed for the direction $\omega = \pi$ as follows:

$$\psi(\rho, \theta, \pi) = \psi(a, \theta, \pi) \, . \tag{29}$$

Substituting from Eq. (27) into Eq. (29), we obtain

$$\psi(\rho, \theta, \omega) = 2\pi \, \delta(\omega - \pi) \quad , \text{for } \omega \text{ about } \pi, \tag{30}$$

which is incorrect. To understand what has gone wrong, we consider an incident angular flux distribution that is not a delta-function, but nonetheless can become a delta-function in a certain limit. In particular, we assume that

$$\psi(a,\theta,\omega) = \frac{1}{\Delta\omega} \quad , \text{ for } \omega \in [\pi - \tfrac{\Delta\omega}{2}, \pi + \tfrac{\Delta\omega}{2}], \text{ and all } \theta. \tag{31}$$

The idea is compute the solution for a "small" value of $\Delta\omega$ and then take the limit of the solution as $\Delta\omega \to 0$. We next consider the geometry illustrated in Fig. 3. Our purpose is to calculate the *incoming* flux at point 2 due to the incident flux at point 1. Note that point 1 is at radius a while point 2 is at radius ρ. Following the integral transport equation, we find that

$$\psi_2^{in}(\omega_2) = \psi_1(\omega_1)\,. \tag{32}$$

The law of sines can be used to relate γ_1 to ω_2, where $\gamma_1 = \pi - \omega_1$:

$$\frac{\rho}{\sin\gamma_1} = \frac{a}{\sin\omega_2}\,. \tag{33a}$$

Since $\sin(\pi - x) = \sin(x)$, for all x, it follows from Eq. (33a) that

$$\frac{\rho}{\sin\gamma_1} = \frac{a}{\sin\gamma_2}\,, \tag{33b}$$

where $\gamma_2 = \pi - \omega_2$. If $\Delta\omega$ is small, then γ_1 and γ_2 are small, and Eq. (33b) yields

$$\gamma_2 = \frac{a}{\rho}(\gamma_1)\,. \tag{34}$$

The contribution to ψ_2^{in} from ψ_1 will be non-zero only if $\gamma_1 \in [-\frac{\Delta\omega}{2}, +\frac{\Delta\omega}{2}]$. Let θ^- denote the location on the surface of the disk for point 1 such that $\gamma_1 = -\frac{\Delta\omega}{2}$, and let θ^+ denote the location on the surface of the disk for point 1 such that $\gamma_1 = +\frac{\Delta\omega}{2}$. These values of θ are illustrated in Fig. 4. The incoming flux at point 2 arises entirely from the band of incident fluxes defined by $\theta^+ \leq \theta \leq \theta^-$. The incident flux at each value of θ within the band corresponds to a unique value of γ_1 that lies within the interval, $[-\frac{\Delta\omega}{2}, +\frac{\Delta\omega}{2}]$; and the incident flux value of $\frac{1}{\Delta\omega}$ in that unique direction generates an angular flux value of $\frac{1}{\Delta\omega}$ at point 2 in the direction corresponding to $\gamma_2 = \frac{a}{\rho}(\gamma_1)$, or equivalently, $\omega_2 = \pi - \frac{a}{\rho}(\gamma_1)$. Since γ_1 varies across the band from $-\frac{\Delta\omega}{2}$ to $+\frac{\Delta\omega}{2}$, it follows that ω_2 varies from $\pi - \frac{a}{\rho}\frac{\Delta\omega}{2}$ to $\pi + \frac{a}{\rho}\frac{\Delta\omega}{2}$. Thus, the incoming angular flux at point 2 is given by

$$
\begin{aligned}
\psi_2^{in}(\omega) &= \frac{1}{\Delta\omega} \quad \text{, for } \omega \in [\pi - \tfrac{a}{r}\tfrac{\Delta\omega}{2}, \pi + \tfrac{a}{r}\tfrac{\Delta\omega}{2}], \\
&= 0 \quad \text{, otherwise.}
\end{aligned}
\tag{35}
$$

Integrating Eq. (35) over the incoming directions, we obtain the "incoming" scalar flux:

$$
\phi_2^{in} = \frac{a}{r} \,. \tag{36}
$$

Taking the limit as $\Delta\omega \to 0$, Eq. (35) becomes

$$
\psi_2^{in}(\omega) = \frac{a}{\rho}\delta(\omega - \pi) \,, \tag{37}
$$

which is in agreement with the correct solution given by Eq. (28).

Thus we see that, for any small but non-zero value of $\Delta\omega$, the angular flux along any characteristic is indeed constant and equal to the incident angular flux along that characteristic. However, in the limit as $\Delta\omega \to 0$, the solution along the characteristic becomes a delta-function scaled by the factor $\frac{a}{r}$. This scaling is actually associated with the scalar flux rather than the angular flux, and arises from the fact that a band of incident fluxes having an angular width of $\Delta\omega$ generates a solution at each interior point with an angular width of $\frac{a}{r}\Delta\omega$. Our notation for the delta-function angular flux solution implies that the angular flux has an r^{-1} dependence because we scale the delta-function by $\frac{a}{r}$. However, this scaling is only done to obtain the correct scalar flux. Because the angular flux is singular, it is only the scalar flux that has meaning. This is why we say that the delta-function angular flux solution "appears" to be non-constant along the characteristic.

This is a facinating result. The solution along a characteristic generated by an incident delta-function angular flux depends upon both the local geometric properties of the surface on which the incident fluxes lie. For instance, if the incident angular fluxes were distributed along two lines, as shown in Fig. 5, rather than on the surface of the disk, as shown in Fig. 1, there would be no radial scaling and Eq. (30) would give the correct solution for $\theta = 0$.

8 Summary

We have shown that one can obtain incorrect solutions by directly inserting incident fluxes with angular delta-function dependencies into the integral transport equation. To avoid this difficulty, one must first obtain the solution for a non-singular approximation to a delta-function incident flux, and then take the limit of that solution as the approximation approaches a delta-function. Alternatively, if one can obtain a solution as a function of a non-singular incident flux with an arbitrary angular dependence, one can simply substitute a delta-function angular dependence into the solution.

We note that our results indicate that existing discrete-ordinates codes should not be used to calculate solutions for a problem with a delta-function angular flux in a “starting” direction. The ‘starting” fluxes are assumed to have zero angular weight, and consequently, are assumed to satisfy a rectilinear equation like Eq. (18a) rather than a curvilinear equation like Eq. (16). However, it is permissible to use a discrete ordinates code to provide an approximate solution for an incident delta-function in any non-starting quadrature direction. The best approach for all problems with incident angular delta-functions is to calculate the uncollided flux analytically, and use the discrete-ordinates code to calculate only the collided component of the flux solution. We stress that this approach is appropriate even for the case of an incident angular delta-function in a starting direction.

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Appendix

The purpose of this Appendix is to describe one-dimensional analytic solutions in both spherical and cylindrical geometries for homogeneous purely absorbing systems with a non-singular but otherwise arbitrary incident flux, and to use these solutions to obtain analogous solutions for delta-function incident fluxes. The incident delta-function fluxes appearing in this Appendix have a more general nature than those appearing in the text, thus we do not normalize these incident fluxes to obtain a unit incident current.

One-Dimensional Spheres

In a homogeneous, purely absorbing medium without sources, the angular flux due to an incoming flux on the boundary is a generalization of Eq. (9):

$$\psi(\overrightarrow{r}, \overrightarrow{\Omega}) = \psi(\overrightarrow{r} - s_b \overrightarrow{\Omega}, \overrightarrow{\Omega}) \exp(-\sigma_a s_b) \, , \tag{38}$$

where σ_a is the absorption cross-section.

In a one-dimensional sphere, $\overrightarrow{r} \cdot \overrightarrow{\Omega} = r\cos\theta = r\mu$, and the angular flux does not depend upon the azimuthal (about the radius vector) angular variable. The geometry and variables used in our derivation are given in Figs (6) and (7) for the cases corresponding to $\mu < 0$ and $\mu > 0$, respectively. The direction, $\overrightarrow{\Omega}$, traverses the sphere of radius a from the point B through point P, which is a distance r from the center at O. It can be seen from the geometry of the figure that

$$\begin{aligned} s_b &= r\cos\theta + \sqrt{a^2 - r^2\sin^2\theta}\,, \\ &= r\mu + \sqrt{a^2 - r^2(1-\mu^2)}\,. \end{aligned} \tag{39}$$

In this expression, the square root is the distance from B to Q, and the other term adds or subtracts depending on whether the cosine is positive or negative. Also, it can be seen from the figure that

$$a\sin\theta' = r\sin\theta\,, \tag{40}$$

from which it follows that

$$\mu' = -\sqrt{1 - (r^2/a^2)(1-\mu^2)}\,, \tag{41}$$

where $\mu' = \cos\theta'$. The expression above states that at a particular position and direction within the sphere, the only contribution to the angular flux comes from an incoming flux on the boundary with a direction cosine equal to μ'. So, if the incoming flux on the boundary is given by an arbitrary distribution, $\psi_b(\mu)$, for $-1 \le \mu \le 0$, then the angular flux in a purely absorbing sphere is given by

$$\psi(r,\mu) = \exp\left\{-\sigma_a\left[r\mu + \sqrt{a^2 - r^2(1-\mu^2)}\right]\right\}\psi_b(\mu')\,, \tag{42}$$

where μ' is given by Eq. (41). This expression satisfies the 1-D spherical-geometry transport equation and the boundary condition, $\psi(a,\mu) = \psi_b(\mu)$, for $-1 \le \mu \le 0$.

Now suppose the incoming flux is in a single direction, that is

$$\psi_b(\mu) = \delta\left(\mu - \mu_1\right) , \quad \text{for } -1 \leq \mu_1 \leq 0, \tag{43}$$

then

$$\psi_b(\mu') = \delta\left[-\mu_1 - \sqrt{1 - (r^2/a^2)\,(1-\mu^2)}\right] . \tag{44}$$

The argument of the above delta-function is zero whenever

$$\mu = \pm\sqrt{1 - (a^2/r^2)\,(1-\mu_1^2)} \equiv \pm\mu_0 \, . \tag{45}$$

However, μ_0 is real only when $r \geq a\sqrt{1-\mu_1^2}$. This results from the fact that the incoming boundary flux is not seen at $r < a\sqrt{1-\mu_1^2}$. Following Eq. (24), we find that

$$\delta\left[-\mu_1 - \sqrt{1-(r^2/a^2)\,(1-\mu^2)}\right] = \frac{|\mu_1|}{|\mu_0|}\frac{a^2}{r^2}\left[\delta(\mu-\mu_0) + \delta(\mu+\mu_0)\right] . \tag{46}$$

It follows from Eqs. (42), (44), and (46), that

$$\psi(r,\mu) = \frac{|\mu_1|}{|\mu_0|}\frac{a^2}{r^2}\left\{\exp\left[-\sigma_a(a\mu_1 + r\mu_0)\right]\delta(\mu-\mu_0) + \exp\left[-\sigma_a(a\mu_1 - r\mu_0)\right]\delta(\mu+\mu_0)\right\} , \tag{47}$$

for $a\sqrt{1-\mu_1^2} \leq r \leq a$, and that $\psi(r,\mu) = 0$, otherwise. If $\mu_1 = -1$, then the solution is non-zero throughout the sphere. In particular, for this case,

$$\psi(r,\mu) = \frac{a^2}{r^2}\left\{\exp\left[-\sigma_a(a+r)\right]\delta(\mu-1) + \exp\left[-\sigma_a(a-r)\right]\delta(\mu+1)\right\} . \tag{48}$$

At $r = a$, this flux consists of two rays, one the incoming boundary flux, and the other the outgoing flux that entered from the other side of the sphere and has been attenuated in traversing the sphere. At the other limit, $\mu_1 = 0$, and the angular flux is everywhere zero.

Cylinders

In one-dimensional cylinders, the analysis is very similar to that for one-dimensional spheres, but with important differences. The cylindrical geometry and the associated variables of interest are shown in Fig. 8. The cylinder is assumed to be infinite along the z-axis. The angular flux is characterized by two angular variables. We use an angular coordinate system, (τ, ω), which is illustrated in Fig. 9. For a ray traversing the cylinder along an arbitrary direction, the τ angle does not change, and the distance from the boundary to a point P is denoted by s_b. The projection of this distance into the $x - y$-plane is given by $s_b \sin\tau$. It follows from Fig. 8 that

$$s_b \sin\tau = \rho\cos\omega + \sqrt{a^2 - \rho^2 \sin^2\omega}\,, \tag{49}$$

and

$$\rho\sin\omega = a\sin\omega'\,, \tag{50}$$

where a denotes the outer radius of the cylinder. The angular flux solution is given in terms of the incident flux, $\psi_b(\tau, \omega)$, as follows:

$$\psi(\rho, \tau, \omega) = \exp\left[-\frac{\sigma_a}{\sin\tau}\left(\rho\cos\omega + \sqrt{a^2 - \rho^2\sin^2\omega}\right)\right]\psi_b(\tau', \omega') \tag{51}$$

where σ_a is the absorption cross section, $\tau' = \tau$ and $\omega' = \sin^{-1}(\rho \sin\omega/a)$. For incoming directions, the above expression is equal to the boundary flux at $\rho = a$. It also satisfies the transport equation in one-dimensional geometry, which is given in terms of our coordinates by

$$\sin\tau\cos\omega\frac{\partial\psi}{\partial\rho} - \frac{1}{\rho}\sin\tau\sin\omega\frac{\partial\psi}{\partial\omega} + \sigma_a\psi = 0\,. \tag{52}$$

In these variables, a delta-function boundary flux is given by

$$\psi_b(\tau,\omega) = \delta(\tau-\tau_1)\delta(\omega-\omega_1)\,. \tag{53}$$

Substituting from Eq. (53) into Eq. (51), we obtain

$$\psi(\rho,\tau,\omega) = \exp\left[-A\right]\delta(\tau-\tau_1)\delta\left[\sin^{-1}\left(\frac{\rho}{a}\sin\omega\right) - \omega_1\right]\,, \tag{54}$$

where

$$A = \frac{\sigma_a}{\sin\tau}\left(\rho\cos\omega + \sqrt{a^2 - \rho^2\sin^2\omega}\right)\,. \tag{55}$$

The last delta-function in Eq. (54) has two zeros at

$$\omega_0 = \sin^{-1}\left(a\sin\omega_1/\rho\right)\,, \tag{56}$$

and

$$\omega_2 = \pi - \sin^{-1}\left(a\sin\omega_1/\rho\right)\,, \tag{57}$$

respectively. Neither of these zeros is real unless $\rho \geq a\sin\omega_1$, which means that the solution is zero for all $\rho < a\sin\omega_1$. Following Eq. (24), we obtain

$$\delta\left[\sin^{-1}\left(\frac{\rho}{a}\sin\omega - \omega_1\right)\right] = \frac{a|\cos\omega_1|}{\rho|\cos\omega_0|}\left[\delta(\omega-\omega_0) + \delta(\omega-\omega_2)\right]\,. \tag{58}$$

The common factor appearing in Eq. (58) results because the absolute value of the cosine of both zeros is the same. Substituting from Eq. (53) into Eq. (54), we get the following expression for the angular flux solution

$$\psi(\rho,\tau,\omega) = \delta(\tau-\tau_1)\,\frac{a|\cos\omega_1|}{\rho|\cos\omega_0|}\left\{\,\delta(\omega-\omega_0)\exp\left[-\sigma_a\left(\frac{\rho\cos\omega_0 - a\,cos\omega_1}{\sin\tau_1}\right)\right]+\right.$$

$$\left.\delta(\omega-\omega_2)\exp\left[-\sigma_a\left(\frac{\rho\cos\omega_2 - a\,cos\omega_1}{\sin\tau_1}\right)\right]\right\} \quad \text{, for } a\sin\omega_1 \le \rho \le a. \tag{59}$$

As previously noted, the angular flux is zero for $0 \le \rho < a\sin\omega_1$. Note that the square root appearing in the exponential in Eq. (54) was taken to be negative when we obtained Eq. (59) because the square root represents a positive quantity (the distance between the points B and Q shown in Fig. 8) and $\cos\omega_1$ is negative.

Every incoming direction that points toward the central axis of the cylinder (the z-axis) has a value of π for the coordinate ω, while the value of the coordinate τ is arbitrary. These directions are analogous to the direction $\mu = -1$ in a one-dimensional sphere because particles in such directions do not change their angular coordinates as they stream, and they pass through the central axis of the cylinder. Restricting ourselves to such directions, we find that $\omega_1 = \pi$, $\omega_0 = 0$, $\omega_2 = \pi$, and the angular flux solution is

$$\psi(\rho,\tau,\omega) = \delta(\tau-\tau_1)\frac{a}{\rho}\left\{\delta(\omega)\exp\left[-\sigma_a(a+\rho)/\sin\tau_1\right]+\right.$$

$$\left.\delta(\omega-\pi)\exp\left[-\sigma_a(a-\rho)/\sin\tau_1\right]\right\} \quad \text{, for } 0 \le \rho \le a. \tag{60}$$

If $\tau_1 = \pi/2$, that is, if the incoming direction is normal to the outer surface of the cylinder, the solution is

$$\psi(\rho, \tau, \omega) = \delta(\tau - \pi/2)\frac{a}{\rho}\left\{\delta(\omega)\exp\left[-\sigma_a(a+\rho)\right] + \right.$$

$$\left. \delta(\omega - \pi)\exp\left[-\sigma_a(a-\rho)\right]\right\} \quad \text{, for } 0 \le \rho \le a. \tag{61}$$

References

1. K. D. Lathrop, “A Comparison of Angular Difference Schemes for One-Dimensional Spherical Geometry S_N Equations,” *Nucl. Sci. Eng.*, **134**, 239 (2000).

2. E. E. Lewis and W. F. Miller, Jr., *Computational Methods of Neutron Transport*, American Nuclear Society, Inc., Lagrange Park, Illinois (1993).

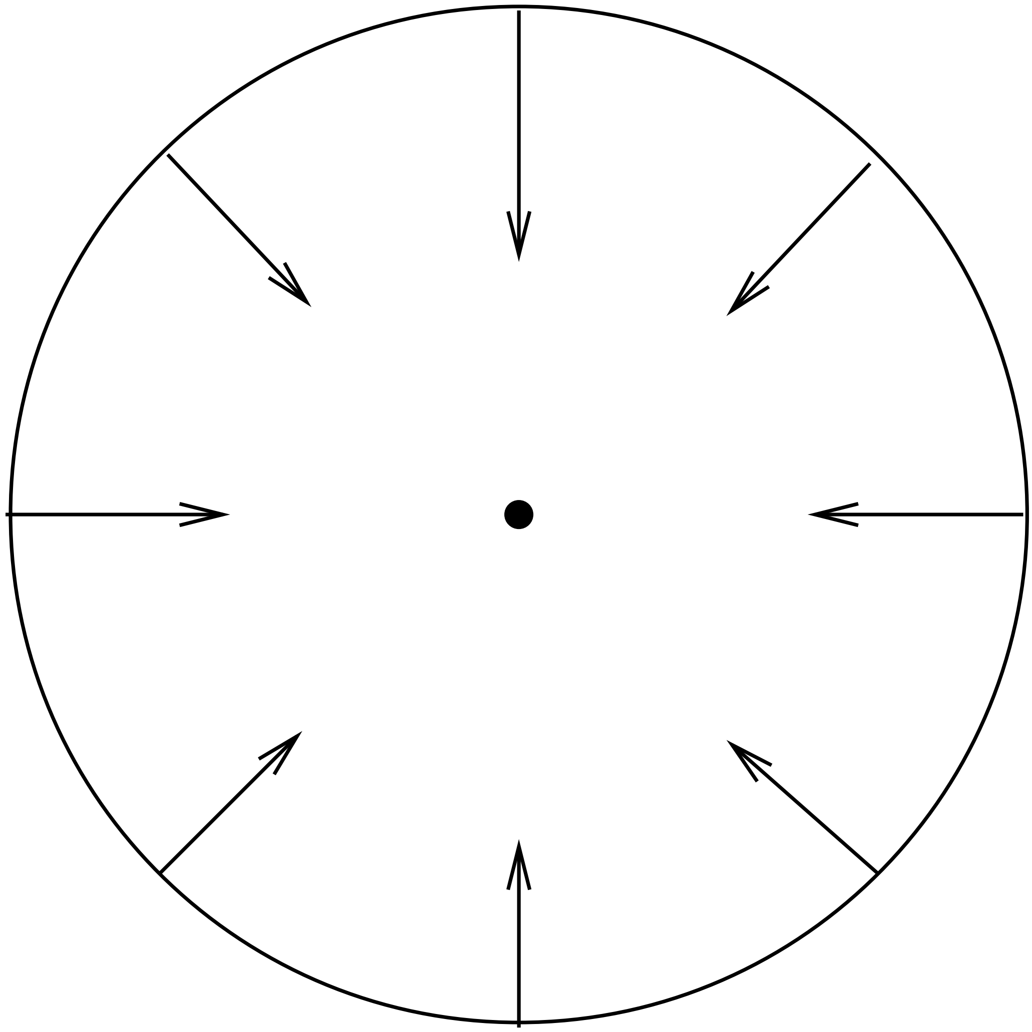

Figure 1: Incident fluxes.

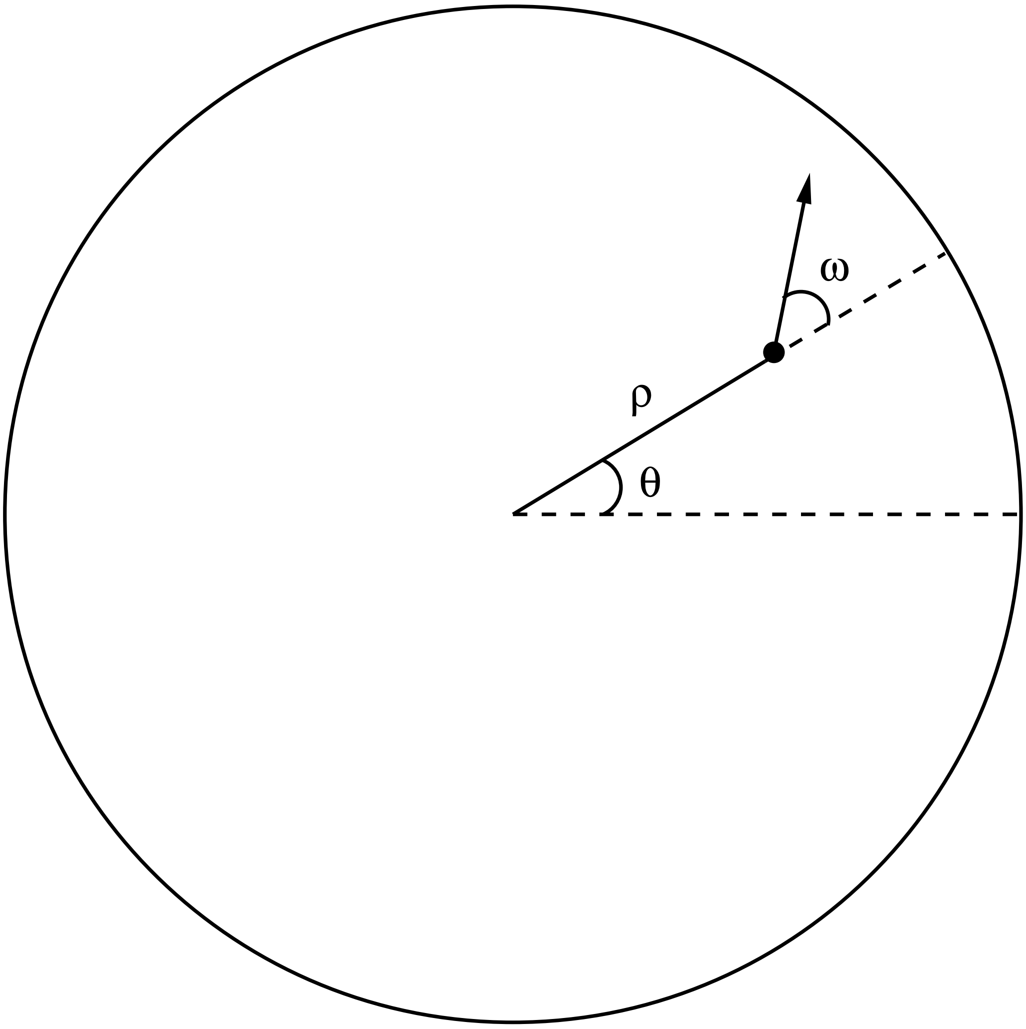

Figure 2: Spatial and directional coordinates.

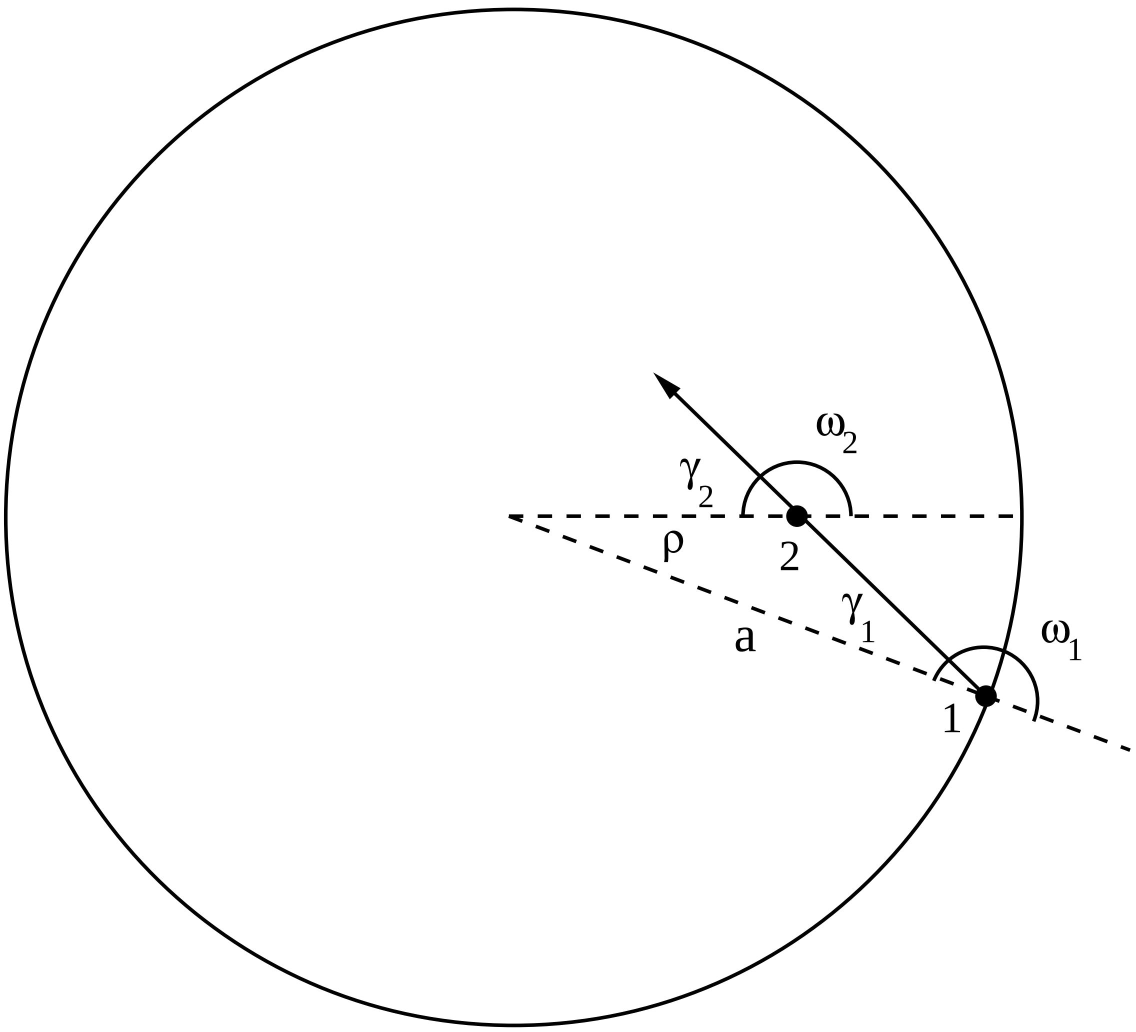

Figure 3: Geometry for “near” delta-function incident fluxes. Note that $\gamma = \pi - \omega$.

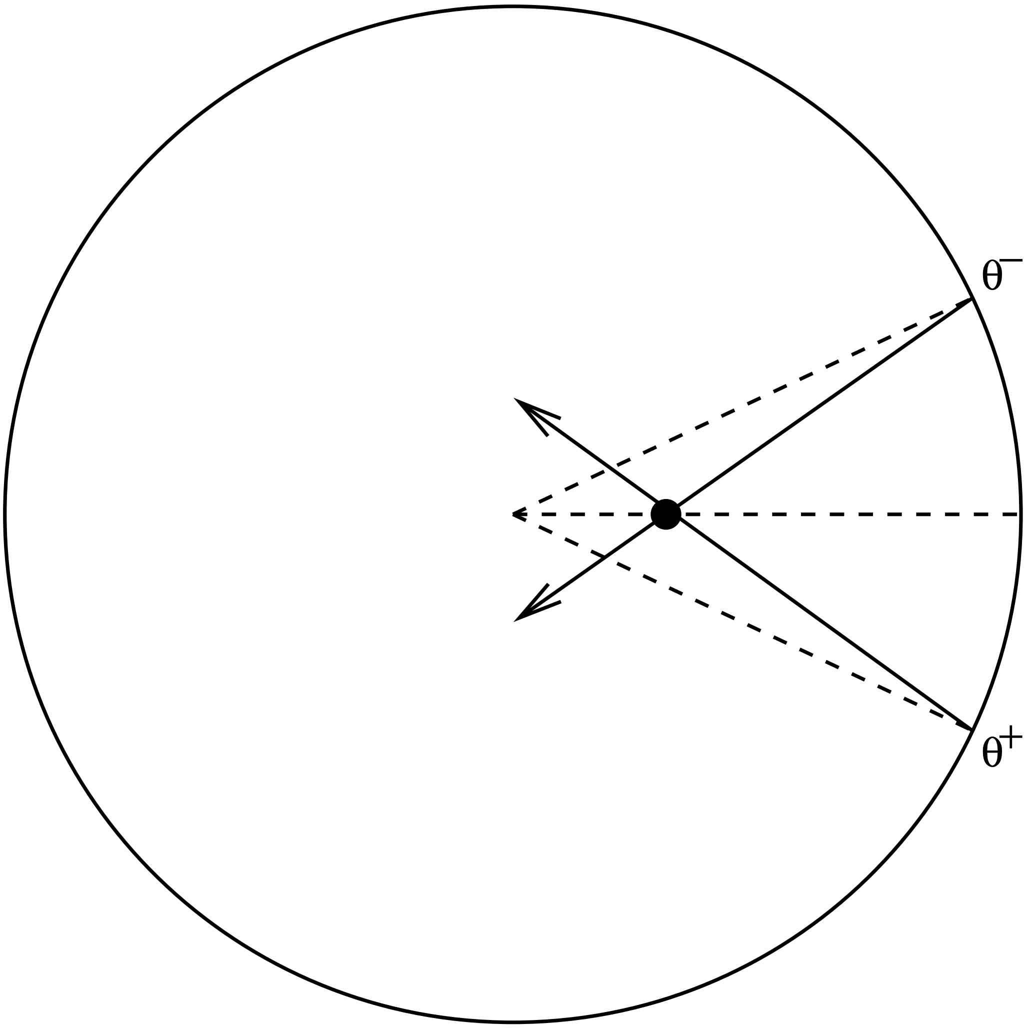

Figure 4: The locations of θ^- and θ^+.

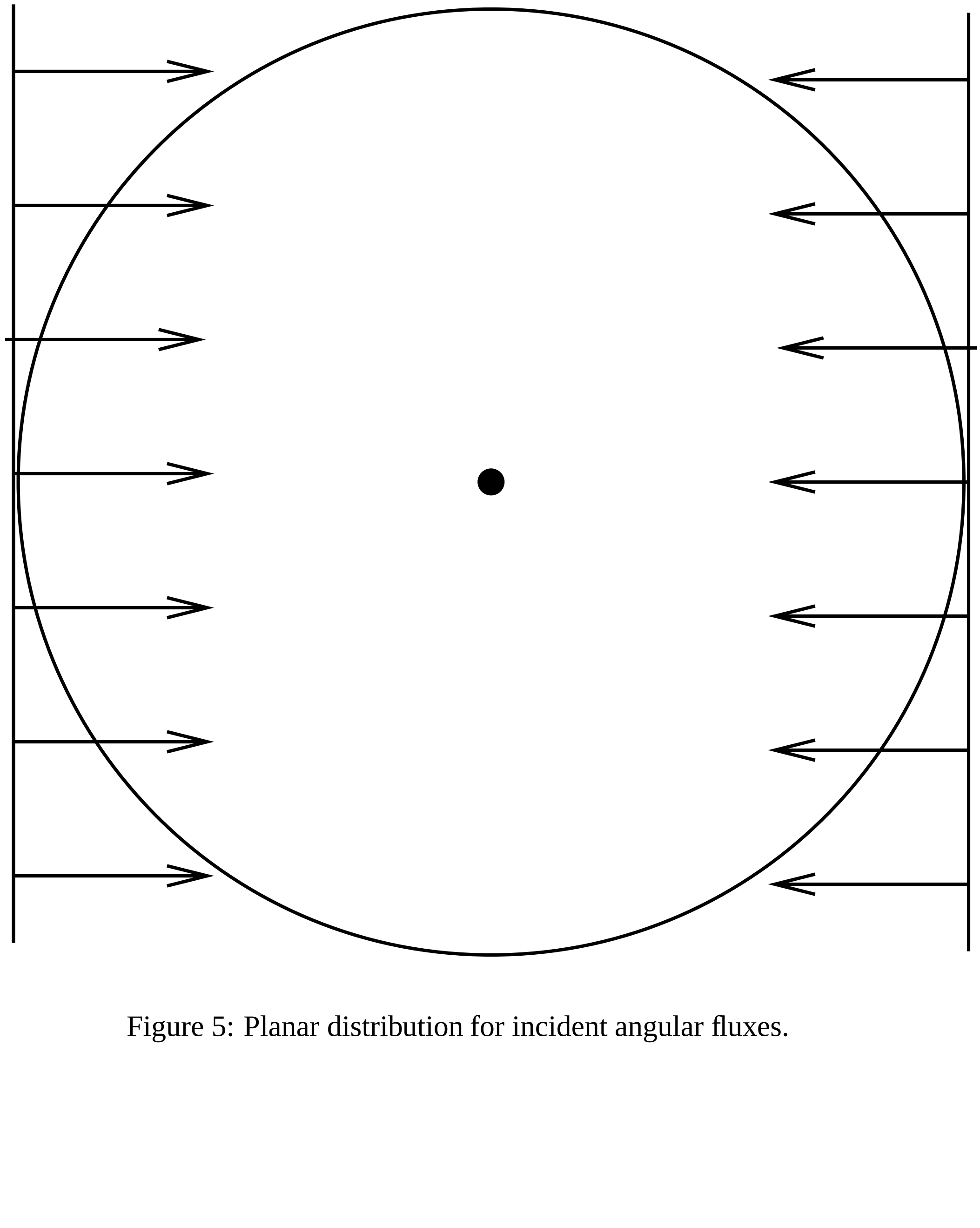

Figure 5: Planar distribution for incident angular fluxes.

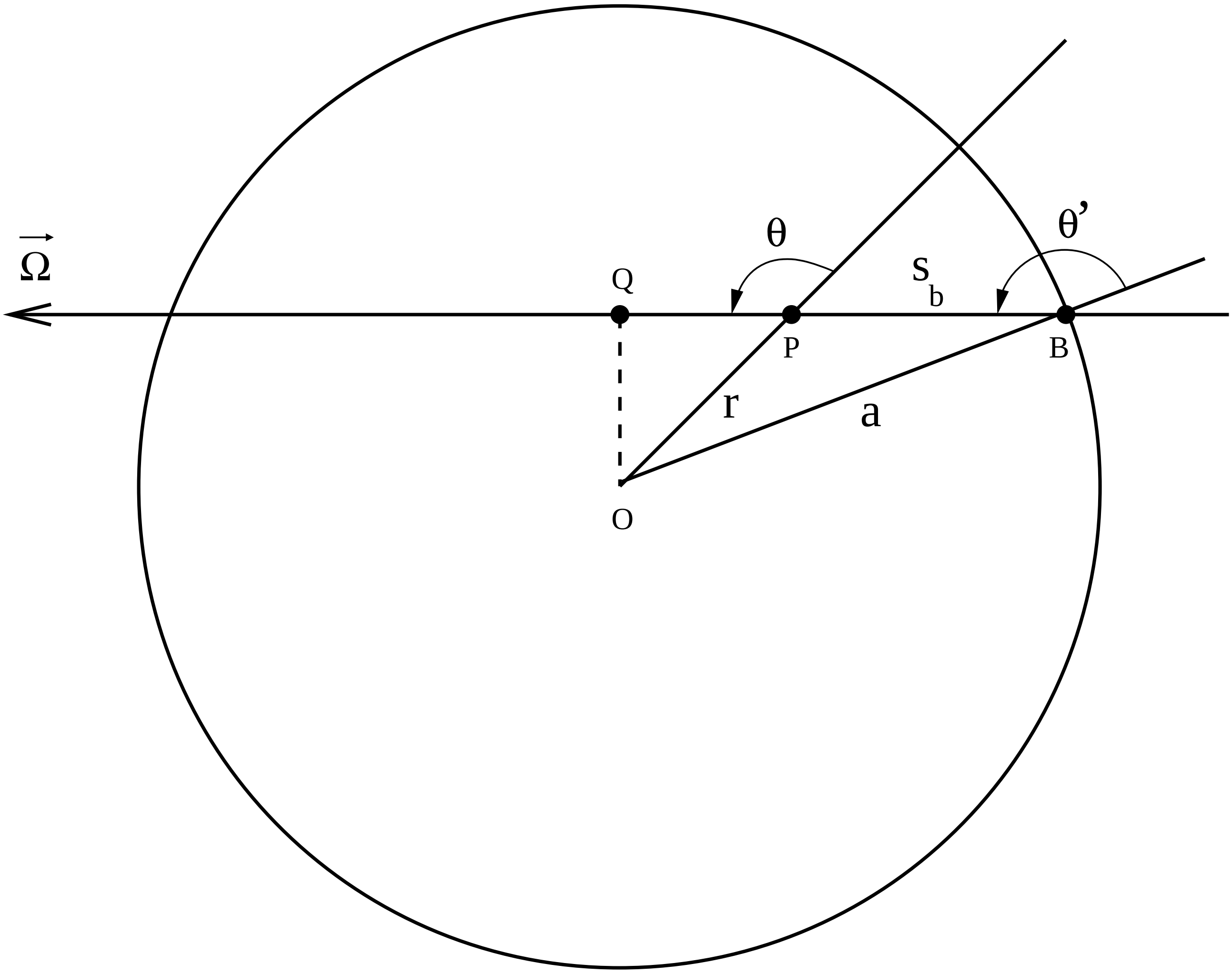

Figure 6: Variables for derivation of 1-D spherical solution, $\mu < 0$.

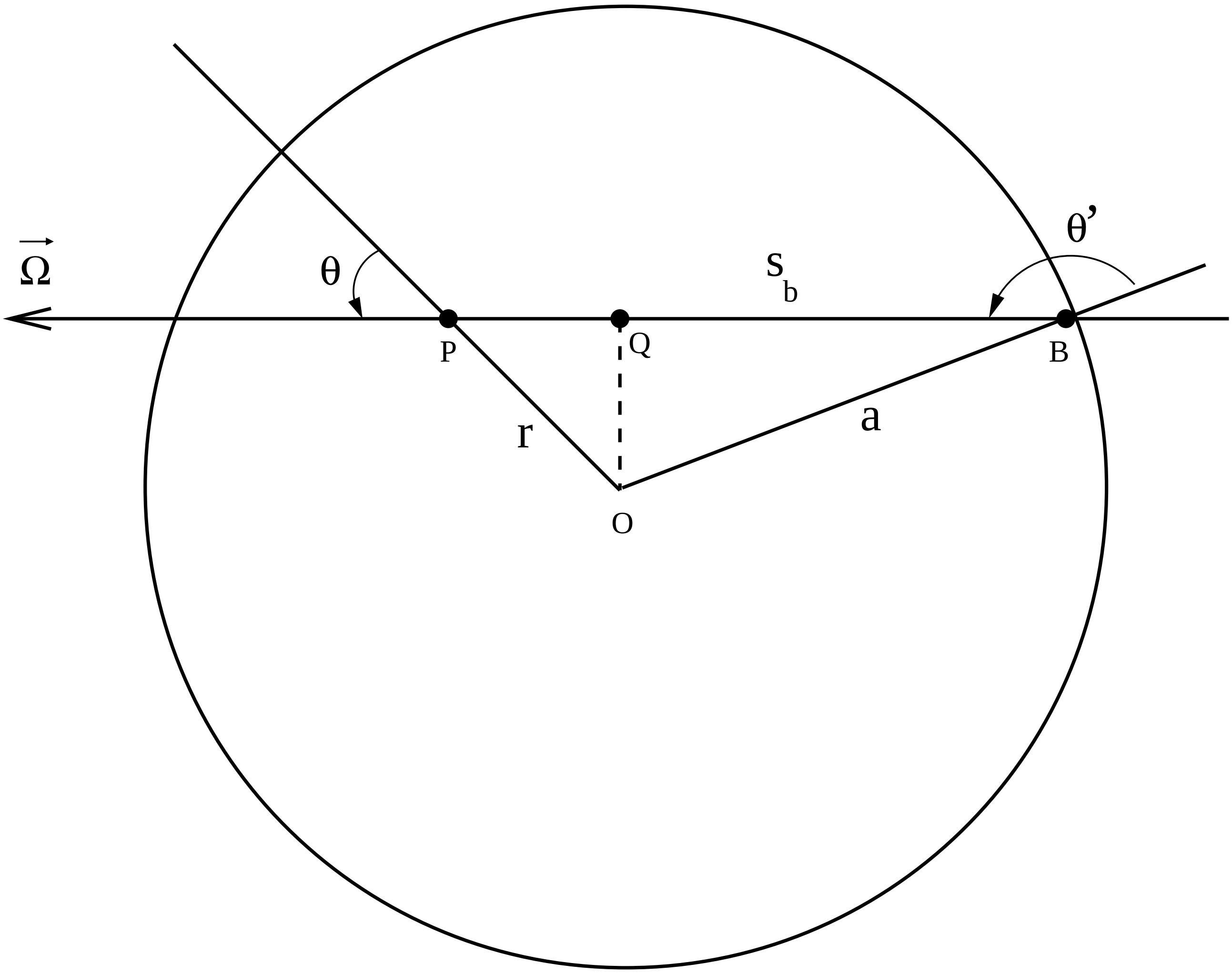

Figure 7: Variables for derivation of 1-D spherical solution, $\mu > 0$.

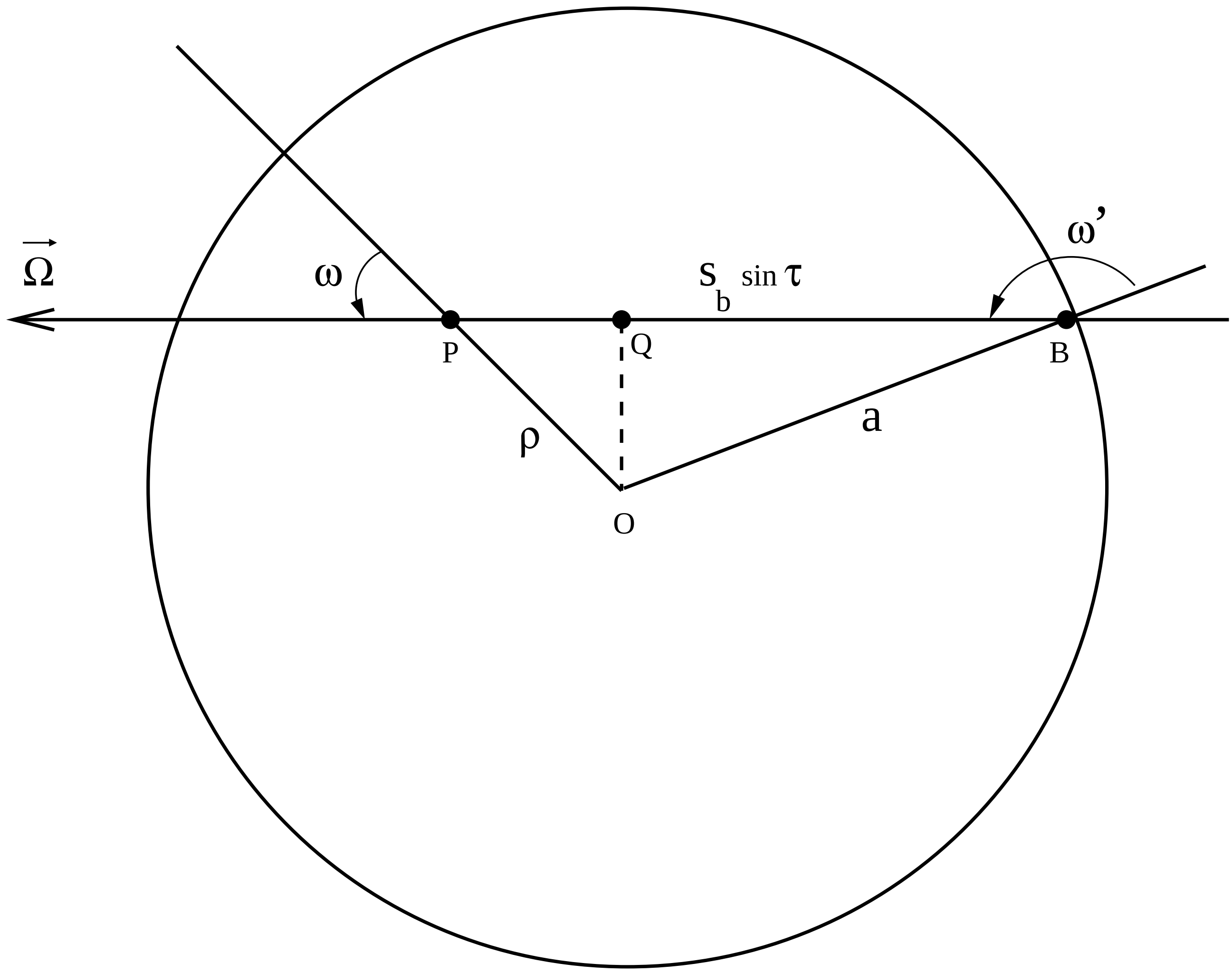

Figure 8: Variables for derivation of 1-D cylindrical solution.

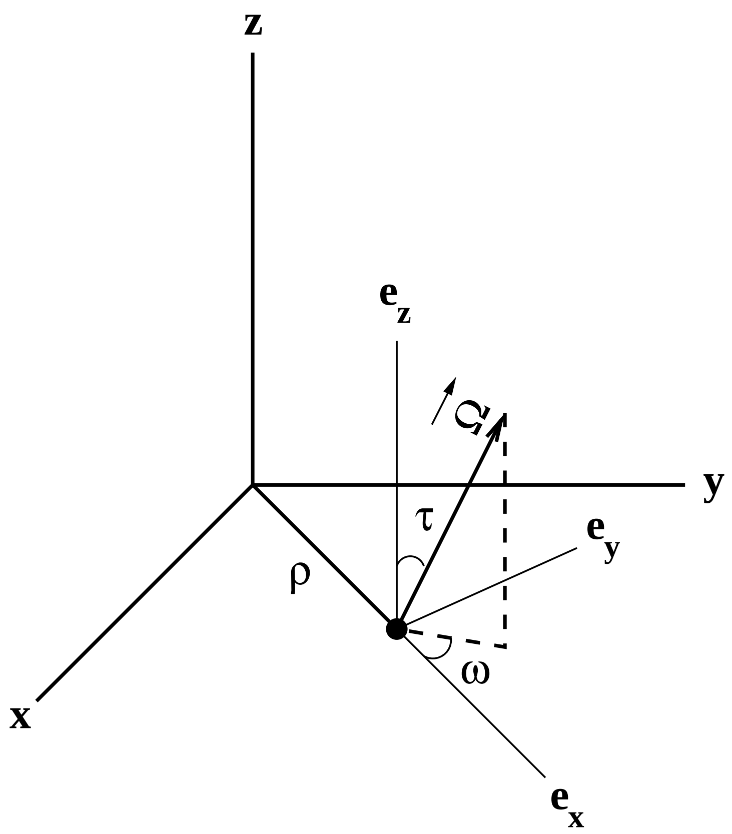

Figure 9: Direction variables for 1-D cylindrical geometry.